Stokes' Theorem

Statement: If $\vec{F}$ is any continuously differentiable vector point function and S is any surface bounded by a curve C, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{Curl}\,\vec{F})\cdot \hat{n}\, ds$$

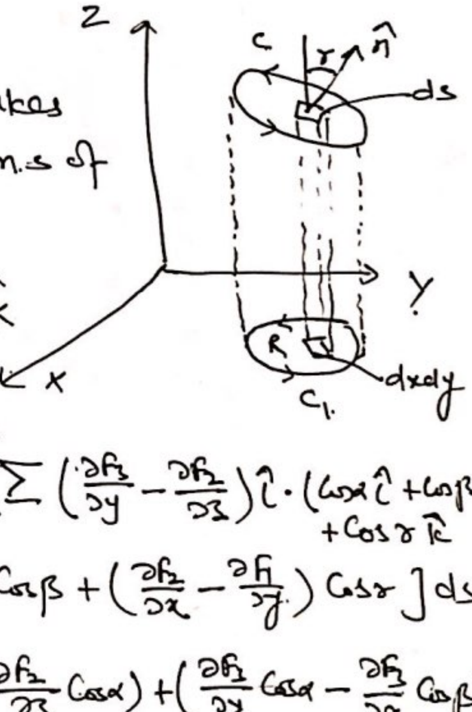
where $\hat{n}$ is unit vector normal to surface S at any pt. and direction of $\hat{n}$ is determined by right hand thumb rule, when rotated in the sense of description of C.

Or

The line integral of the tangential component of a continuously differentiable vector point function over a simple closed curve is equal to the surface integral of the normal component of curl of vector point function over the surface bounded by that curve.

Proof: Let S be a surface which is such that its projection on XY, YZ, ZX planes are regions bounded by simple closed curves and suppose S be represented simultaneously in the forms $z = f(x,y)$, $x = g(x,z)$, $y = h(z,x)$

where f, g, h are single valued, continuous and have cont. first order partial derivatives.



Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and $\hat{n}$ makes angle α, β, γ with the directions of x, y and z axes resp.

$$\therefore\ \hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

The reqd. result may be written

as $$\oint (f_1dx + f_2dy + f_3dz) = \iint_S \left[\Sigma\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i}\cdot(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})\right]ds$$

$$= \iint_S\left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right]ds$$

$$= \iint_S\left[\left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right) + \left(\frac{\partial f_2}{\partial x}\cos\gamma - \frac{\partial f_2}{\partial z}\cos\alpha\right) + \left(\frac{\partial f_3}{\partial y}\cos\alpha - \frac{\partial f_3}{\partial x}\cos\beta\right)\right]ds \quad \text{——(1)}$$

Let R be orthogonal projection of S on xy-plane and let C_1 be its boundary as shown in figure.

If $z = f(x,y)$ be taken as eq of S, then $\hat{n}$ has direction along $\nabla(z - f(x,y)) = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$

$$\therefore \frac{\cos\alpha}{-\frac{\partial f}{\partial x}} = \frac{\cos\beta}{-\frac{\partial f}{\partial y}} = \frac{\cos\gamma}{1} \quad \text{——— (2)}$$

Now $\oint_C f_1 dx = \oint_{C_1} f_1(x,y,f(x,y))\,dx$

$$= \oint [f_1(x,y,f(x,y))dx + 0\,dy] = \iint_R \left[\frac{\partial}{\partial x}(0) - \frac{\partial f_1}{\partial y}\right] dx\,dy$$

[By Green thm in plane for region R]

$$= -\iint_R \frac{\partial}{\partial y} f_1(x,y,f(x,y))\,dx\,dy$$

$$= -\iint_R \left[\frac{\partial}{\partial y} f_1(x,y,z) + \frac{\partial}{\partial z} f_1(x,y,z)\frac{\partial f}{\partial y}\right] dx\,dy$$

[$\because z = f(x,y)$]

$$= -\iint_R \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\,dy$$

$$= -\iint_S \left[\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial f}{\partial y}\right]\cos\gamma\, ds = -\iint_S \left(\frac{\partial f_1}{\partial y} - \frac{\partial f_1}{\partial z}\frac{\cos\beta}{\cos\gamma}\right)\cos\gamma\, dx\,dy$$

[by (2)]

$$\Rightarrow \oint_C f_1 dx = \iint_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right) ds \quad \text{——— (3)}$$

||ly $\oint_C f_2 dy = \iint_S \left(\frac{\partial f_2}{\partial x}\cos\gamma - \frac{\partial f_2}{\partial z}\cos\alpha\right) ds$ ——— (4)

and $\oint_C f_3 dz = \iint_S \left(\frac{\partial f_3}{\partial y}\cos\alpha - \frac{\partial f_3}{\partial x}\cos\beta\right) ds$ ——— (5)

Adding (3), (4) and (5), we have obtained (1)

i.e. $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\text{Curl}\,\bar{F})\cdot\hat{n}\, ds$

If the surface S does not satisfy the restrictions imposed above, even then Stoke's thm will be true provided S can be subdivided into surfaces $S_1, S_2, ---, S_k$ with boundaries $C_1, C_2, --, C_k$ which do satisfy restrictions.

Stoke th^m holds for such surfaces. The sum of surface integrals over $S_1, S_2, \ldots, S_k$ will give us surface integral over S while the sum of the integrals over $C_1, C_2, \ldots, C_k$ will give us line integral over C.

Q: Prove that Green th^m in plane is a particular case of Stoke's th^m.

Sol^n: Let the given surface S lie in xy-plane bounded by closed curve C, then unit normal vector $\hat{n}$ to surface S lies along z-axis.

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_C (F_1 dx + F_2 dy) \quad \text{——} (1)$$

$$\text{Now} \quad \text{Curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\therefore \iint_S \text{Curl}\,\bar{F}\cdot\hat{n}\, ds = \iint_S \text{Curl}\,\bar{F}\cdot\hat{k}\, ds = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \quad \text{——} (2)$$

where $ds = dx\,dy$ = elementary area of S which lie in xy-plane

By Stoke th^m $\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{Curl}\,\bar{F}\cdot\hat{n}\, ds$

$$\Rightarrow \oint_C (F_1 dx + F_2 dy) = \iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Which is Green's th^m in plane (by (1) and (2))

Q: If S is an open surface bounded by a closed curve C, then show that (7)

(i) $\oint_C \vec{r}\cdot\vec{dr} = 0$ (ii) $\oint_C \phi\nabla\phi\cdot\vec{dr} = 0$

(iii) $\oint_C \phi\nabla\psi\cdot\vec{dr} = -\oint_C \psi\nabla\phi\cdot\vec{dr}$

Solⁿ (i) $\oint_C \vec{r}\cdot\vec{dr} = \iint_S (\nabla\times\vec{r})\cdot\hat{n}\,ds$ [By Stokes' thᵐ]

$= \iint_S \vec{0}\cdot\hat{n}\,ds = 0$ [$\because \nabla\times\vec{r} = \vec{0}$].

(ii) $\oint_C \phi\nabla\phi\cdot\vec{dr} = \iint_S \nabla\times(\phi\nabla\phi)\cdot\hat{n}\,ds$ [By Stokes' thᵐ]

$= \iint_S [\nabla\phi\times\nabla\phi + \phi\nabla\times(\nabla\phi)]\cdot\hat{n}\,ds$

$= \iint_S [\vec{0}+\phi\vec{0}]\cdot\hat{n}\,ds = 0$

(iii) By Stokes' thᵐ, we have

$\oint_C \vec{F}\cdot\vec{dr} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds$

Taking $\vec{F} = \nabla(\phi\psi)$, we get

$\oint_C \nabla(\phi\psi)\cdot\vec{dr} = \iint_S [\text{Curl}\,\nabla(\phi\psi)]\cdot\hat{n}\,ds$

$\Rightarrow \oint_C (\phi\nabla\psi+\psi\nabla\phi)\cdot\vec{dr} = 0$ [$\because$ Curl grad $f = \vec{0}$]

$\Rightarrow \oint_C \phi\nabla\psi\cdot\vec{dr} + \oint_C \psi\nabla\phi\cdot\vec{dr} = 0$

$\Rightarrow \oint_C \phi\nabla\psi\cdot\vec{dr} = -\oint_C \psi\nabla\phi\cdot\vec{dr}$

Q: Prove that $\oint_C \phi\nabla\psi\cdot\vec{dr} = \iint_S (\nabla\phi\times\nabla\psi)\cdot\hat{n}\,ds$

Solⁿ: By Stoke's thᵐ $\oint_C \vec{F}\cdot\vec{dr} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds$

Taking $\vec{F} = \phi\nabla\psi$, we get

$\oint_C \phi\nabla\psi\cdot\vec{dr} = \iint_S \text{Curl}(\phi\nabla\psi)\cdot\hat{n}\,ds$

$= \iint_S (\nabla\phi\times\nabla\psi + \phi\,\text{Curl}\,\nabla\psi)\cdot\hat{n}\,ds$

[$\because$ Curl grad $\psi = \vec{0}$]

$\Rightarrow \oint_C \phi\nabla\psi\cdot\vec{dr} = \iint_S (\nabla\phi\times\nabla\psi)\cdot\hat{n}\,ds$
